Calculation of odds ratios and confidence intervals for link functions

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Odds ratios for ordinal data

Logit odds ratio

Let $p_h = P(Y = h), h = 1, 2, \ldots, ncat$. For the logit link

$$\log\left(\frac{p_h}{1-p_h}\right) = \delta_h + \beta x$$

By exponentation of both sides, this becomes

$$\frac{p_h}{1-p_h} = \exp(\delta_h + \beta x)$$

Consider the cases where *x* = 0 and *x* = 1. Denote the probabilities for the two cases by p_{0h} and p_{1h} respectively.

For *x* = 1

$$\frac{p_{1h}}{1-p_{1h}} = \exp(\delta_h + \beta) \quad (1)$$

and for *x* = 0

$$\frac{p_{0h}}{1-p_{0h}} = \exp(\delta_h) \quad (2)$$

The odds ratio can be expressed in terms of (1) and (2) as

$$\frac{\dfrac{p_{1h}}{1-p_{1h}}}{\dfrac{p_{0h}}{1-p_{0h}}} = \frac{\exp(\delta_h + \beta)}{\exp(\delta_h)} = \exp(\beta) = f\left(\beta\right) \tag{3}$$

The odds ratio thus reduces to $\exp(\beta)$. The expressions above are also valid for a non-binary x : in all cases, it is simply the effect of a unit increase in *x*.

CLL odds ratio

Let $p_h = P(Y = h), h = 1, 2, ..., ncat$. For the CLL link, consider

$$\ln[-\ln(1-p_h)] = \delta_h + \beta x \tag{4}$$

where δ_h is the threshold for category *h*. Therefore

$$\ln(1-p_h) = -\exp(\delta_h + \beta x) \tag{5}$$

Consider the cases where *x* = 0 and *x* = 1 and let $\phi_x = \exp(\delta_h + \beta_x)$. Thus

$$\phi_1 = \exp(\delta_h + \beta), \phi_0 = \exp(\delta_h).$$

From (5) it follows that

$$1 - p_{1h} = \exp(-\phi_1); \quad p_{1h} = 1 - \exp(-\phi_1).$$

The odds (*x* = 1) is

$$\begin{aligned} \frac{p_{1h}}{1-p_{1h}} &= \frac{1-\exp(-\phi_1)}{\exp(-\phi_1)} \\ &= \exp(\phi_1) - 1. \end{aligned}$$

Likewise the odds (*x* = 0) is

$$\frac{p_{0h}}{1-p_{0h}} = \exp(\phi_0) - 1.$$

The CLL ratio is therefore

$$\frac{\exp(\phi_1) - 1}{\exp(\phi_0) - 1} = f(\beta)$$

which simplifies to

$$f(\beta) = \frac{\exp[\exp(\beta)] - \exp[-\exp(\delta_h)]}{1 - \exp[-\exp(\delta_h)]}. \tag{6}$$

Probit odds ratio

Let $p_{1h} = \phi(\delta_h + \beta)$ where ϕ denotes the CDF of the $N(0,1)$ distribution. Similarly, let $p_{0h} = \phi(\delta_h)$ when β = 0.

The odds ratio is thus

$$\frac{p_{1h}(1 - p_{1h})}{p_{0h}(1 - p_{0h})} = f(\beta). \tag{7}$$

Log-log odds ratio

Similarly, for the log-log link, let

$$\phi_x = \exp[-(\delta_h + \beta x)], \; h = 1, 2, ..., ncat1 \, . \tag{8}$$

In this case

$$p_{1h} = \exp(-\phi_1), \quad x = 1$$

and

$$p_{0h} = \exp(-\phi_0), \quad x = 0.$$

The odds ratio is then

$$\frac{p_{1h}(1-p_{0h})}{p_{0h}(1-p_{1h})} = \frac{\exp(-\phi_1)[1-\exp(-\phi_0)]}{\exp(-\phi_0)[1-\exp(-\phi_1)]}$$

which simplifies to

$$f(\beta) = \frac{\exp(\phi_0)-1}{\exp(\phi_1)-1} \tag{9}$$

95% confidence intervals

Define the odds ratio as $f(\beta)$ and denote $E(\beta)$ by μ_β . A first-order Taylor expansion gives

$$f(\beta) \simeq f(\mu_\beta) + (\beta - \mu_\beta) f^{'}(\mu_\beta \mid \beta = \mu_\beta) \tag{10}$$

An approximate expression for the first-order derivative is $f^{'}(\beta) \simeq [f(\beta+\varepsilon) - f(\beta)]/\varepsilon$, where $\varepsilon = 0.000001.$

Thus

$$\text{var}(f(\beta)) \simeq \text{var}(\beta) \times (f^{'})^2.$$

Denote the standard deviation of $f(\beta)$ by $\sigma_{f(\beta)}$, then

$$\sigma_{f(\beta)} \simeq \sigma_\beta \times abs(f^{'}).$$

The approximate 95% confidence interval is then

$$f(\beta) \pm 1.96\sigma_{f(\beta)}. \tag{11}$$

Odds ratios for binary data

The derivation of the odds ratio for the Bernoulli and binomial models are obtained as shown above by replacing δ_h with β_0, where β_0 denotes the intercept. The odds ratio of β_0 is obtained by using the term $\beta_0 x$ where *x* is equal to 1 or 0.