Method for estimation of level-1 variance for log links

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For the logit, probit, log-log and complimentary log-log link functions, it is possible to produce intraclass correlation coefficients since the level-1 variances are assumed to be $\pi^2/3$, 1, $\pi^2/6$, and $\pi^2/6$ respectively. However, there is no equivalent assumption for the log link. In this document we describe a method that is used by the program for level-1 variation estimation in the case of a log link function.

Consider the model

$$\eta_{ijk} = \mathbf{x}'\boldsymbol{\beta} = u_j + v_k \qquad (1)$$

where $u_j \sim N(0, \phi_{(2)})$ and $v_k \sim N(0, \phi_{(3)})$ where it is assumed that u_j and v_k are independently distributed.

For models based on the log link function the following model is imposed on the means

$$\mu_{ijk} = \exp(\eta_{ijk}) \qquad (2)$$

The model (1) is transformed to a linear model by

$$\eta_{ijk} = \ln(\mu_{ijk}) \qquad (3)$$

Using the following well-known results for moment generating functions (MGF) an estimate of $\mu = \frac{1}{N}\sum \mu_{ijk}$ is derived.

The MGF of a random variable *X* is defined as

$$M(t) = E(\exp(tX)) \quad t = 1, 2, \ldots \qquad (4)$$

If X and Y are independent with MGFs $M_x(t)$ and $M_y(t)$ then the MGF of *X*+*Y* is

$$M_x(t) \cdot M_y(t) \qquad (5)$$

If $a \in \mathbb{R}$ is a constant, then the MGF of $a + X$ is

$$\exp(ta) \cdot M_x(t) \tag{6}$$

If $X \sim N(0, \sigma^2)$ then

$$M(tX) = \exp(t\frac{\sigma^2}{2})$$

For $t = 1$

$$E(\exp(X)) = \exp\frac{\sigma^2}{2} \tag{7}$$

Using results (5) to (7) and by letting $a_{ijk} = \mathbf{x}_{ijk}^{'} \hat{\boldsymbol{\beta}}$ it follows that

$$E(\mu_{ijk}) = \exp(a_{ijk} + \frac{1}{2}\hat{\phi}_{(2)} + \frac{1}{2}\hat{\phi}_{(3)}) \tag{8}$$

Hence

$$\hat{\mu} = \frac{1}{N}\sum_{i,j,k}(E(\mu_{ijk})) \tag{9}$$

We use this value of $\hat{\mu}$ when estimating the level-1 variances of the Poisson, Negative Binomial, Gamma and Inverse Gaussian distributions for the purpose of calculating intraclass correlation coefficients. Note that the variances of these distributions are all functions of μ:

Poisson: $\sigma^2 = \mu$
Negative binomial: $\mu + \psi\mu^2$
Gamma: $\psi\mu^2$
Inverse Gaussian: $\psi\mu^3$